Periodicity and growth in a lattice gas with dynamical geometry

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We study a one-dimensional lattice gas "dynamical geometry model" in which local reversible interactions of counter-rotating groups of particles on a ring can create or destroy lattice sites. We exhibit many periodic orbits and show that all other solutions have asymptotically growing lattice length in both directions of time. We explain why the length grows as $\sqrt{t}$ in all cases examined. We completely solve the dynamics for small numbers of particles with arbitrary initial conditions.

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I. INTRODUCTION

Lattice models are ubiquitous in physics, whether as regularizations for continuum theories (quantum field theory, quantum gravity), scaffolding for numerical methods (classical field theories, continuum mechanics), or because the lattice is physically real (condensed matter physics). One may distinguish between applications in which the lattice structure and size are fixed, or at least not dynamical, and those in which the lattice itself evolves dynamically. Examples of dynamical lattices include the causal dynamical triangulation approach to quantum gravity [1], the variable-length lattice models of recent relevance to the AdS/CFT correspondence (the conjectured equivalence between type IIB superstring theory on the product $\mathrm{AdS}_5 \times \mathrm{S}^5$ of the five-dimensional Anti-de Sitter space with the five-dimensional sphere, and the $N=4$ supersymmetric Yang-Mills gauge theory in four dimensions) [2], and models of evolving networks such as the World Wide Web [3].

In Ref. [4], Hasslacher and Meyer constructed a lattice gas model with dynamical geometry and a reversible evolution rule. It can be viewed as a toy model for general relativity in that the geometry (length) of the one-dimensional lattice changes in response to the motion (scattering) of the matter particles on it, although the detailed dynamics is very different. We study it because it is the uniquely simplest one-dimensional reversible lattice gas with dynamical geometry, its classical dynamics may well be exactly soluble, and it should be straightforward to quantize despite the dynamical background geometry. In this paper, we extend the previous analyses of the classical dynamics. The central issue is the long-time behavior of the lattice length. We will completely solve the dynamics for systems of a few particles, and explain the typical $\sqrt{t}$ growth of the length which has been seen previously in simulations.

The model consists of a one-dimensional lattice of L sites with periodic boundary conditions (a ring), where L may change with time. The initial state contains N_R right-moving particles and N_L left-moving ones, which may be placed arbitrarily on the sites subject to an exclusion principle: two particles moving in the same direction may not occupy the same site. The numbers of left and right movers are each conserved during evolution. Time proceeds in discrete steps. At each time step, the particles first *advect*: each particle moves one site in its own direction of motion. Then the particles interact, or *scatter*, according to the following rules (see Fig. 1).

(1) If a right and left mover occupy the same site, this site is replaced by two sites, with the right mover on the right-most site and the left mover on the left-most site.

(2) If two adjacent sites are singly occupied, with a right mover on the right-most site and a left mover on the left most, these sites are replaced by a single site occupied by both particles.

(2a) Although this is not an independent rule, we emphasize that rule (2) is not applied to doubly occupied sites since the resulting state would violate the exclusion principle.

The scattering rules are applied simultaneously to all lattice sites, in parallel rather than sequential update. These rules define a reversible dynamics in that every state has not only a unique successor but also a unique predecessor.

Point (2a), which we will sometimes refer to as the "exclusion rule," is the major complication in analyzing the dynamics, as will be seen in Sec. II below. The situation it describes can arise following advection when a single particle moving in one direction approaches a pair of particles on adjacent sites moving in the other. Such a pair of particles moving in the same direction on adjacent sites will be called simply a *pair*, and we will see that such "bound states" can play the role of quasiparticles in the system.

Example 1: One against one. As an illustration of the dynamics following from these rules, consider the trivial case of a single right mover facing a single left mover on a

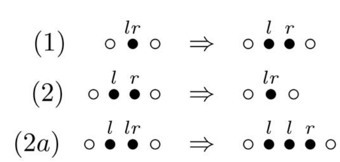

FIG. 1. The scattering rules. Filled-in sites are occupied. Here l denotes a left mover, r a right mover.

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lattice of length $L>2$. The particles scatter twice as each makes a complete circuit around the lattice, and the exclusion rule obviously cannot apply. Let $d(r,l)$ be the length of the arc from the right mover to the left mover which is shrinking during advection, and $p=|d(r,l)|$ its parity mod 2. The next interaction will create or destroy a site according as p is 0 or 1. We display the pair $[p;|L|]$ following each scattering, where $|L|$ is the parity of the lattice length. These parities change only during scattering, not advection, and suffice to determine the sequence of site creation/destruction events. To the right we give the value of the growth ΔL resulting from each interaction:

$$\begin{array}{ll} [1;1] & \\ & \downarrow -1 \\ [0;0] & \\ & \downarrow +1 \\ [0;1] & \\ & \downarrow +1 \\ [1;0] & \\ & \downarrow -1 \\ [1;1] & \end{array}$$

The system returns to its initial state, with the same lattice length, after four scattering events. All four possible parity combinations of $[p;|L|]$ appear in the above cycle, and any one can be viewed as the "initial" state. The evolution in this simplest case is periodic in time independent of the initial values of L and $d(r,l)$.

This example suggests, and Sec. II will confirm, the importance of parity for this system. In general, one might expect that each right mover will interact with each left mover as they pass. Each interaction changes the parity of the length L, since $\Delta L=\pm 1$. The changes in parity would thus be highly predictable. The importance of the exclusion rule is that the occurrence of pairs spoils this predictability. For example, when a right mover faces a pair of left movers, it may eventually land on the leading or the trailing member of the pair. In the former case two interactions occur as expected, with $\Delta L=0$, but in the latter case, shown as (2a) in Fig. 1, only a single interaction occurs and $\Delta L=+1$.

After initial explorations in Ref. [4], numerical simulations of the evolution of initial states rather densely populated with particles (25 particles on a 50 site lattice) were carried out in Ref. [5]. Despite the reversibility of the dynamics, most initial states result in growth of the lattice size, empirically as $L(t)\sim\sqrt{t}$ at late times, with large fluctuations. On small lattices, some "rogue states" were also found with $L(t)$ periodic, but the proportion of these dropped off rapidly with increasing initial lattice size. Two versions of mean-field theory were proposed to explain the observed growth rate, one of which predicted $\sqrt{t}$ growth as observed while the other predicted $t^{1/3}$. In this paper we will identify many periodic solutions on lattices of arbitrary size, and also propose an alternative explanation for the typical $\sqrt{t}$ growth. The reversibility of the dynamics leads to the following simple but fundamental

Evolution theorem. Every solution of the model is either periodic, or grows without bound in both directions of time.

Proof. Consider a solution for which the lattice length remains bounded in one direction of time, say $t\to\infty$. Since there are only finitely many distinct states of this system on a lattice of given size, the evolution must eventually return to some previous state. The evolution is then periodic from this time on. By reversibility and uniqueness, it is periodic in backward time also.

The organization of this paper is as follows. In Sec. II we give a general analysis of the evolution of initial states, assuming that no pairs are present or form later. We emphasize the crucial role of parity in the problem (first noted in Ref. [5]) and establish the existence of many periodic solutions. We explain why growing solutions of this kind must grow as $\sqrt{t}$. In Sec. III (with details in the appendices) we exhaustively analyze the evolution of all initial states containing at most four particles, possibly including pairs. We see cases in which permanent pairs form and behave as quasiparticles or bound states in the system. The major obstacle to a complete solution of this model is the lack of a general framework for describing these quasiparticles and their effects. Section IV contains conclusions and open problems. We would like to acknowledge discussions at the early stages of this work with the authors of Ref. [6], who have independently obtained similar results.

II. GENERAL ANALYSIS OF STATES WITHOUT NEAREST-NEIGHBOR PAIRS

Although simulations of the time evolution of "random" initial states look quite complicated, there are several important general principles governing the dynamics which can be formulated. First, the translational (rotational) symmetry of the lattice allows the dynamics to be viewed in various reference frames. We have thus far used a frame fixed with respect to the lattice, but we can transform to the rest frame of either the right- or the left-moving particles. The rest frame of the left movers, for example, is defined as follows. At each advection step, the left movers move one site to the left. We can follow this advection step with a symmetry (gauge) transformation which rotates every particle one step to the right, thus undoing the advection for the left movers. This is followed by the scattering step as usual. In this frame, or gauge, the left movers do not advect, while the right movers advect *two* sites per time step.

The use of the rest frame for one group of particles makes it clear that parity (mod 2) plays a crucial role in the dynamics. The size of the gap between two left movers, or two right movers, is trivially preserved by advection, but so is the parity of the (shrinking) interval between a left and a right mover. If this interval contains no other particles, its parity determines the character of the eventual interaction between these particles, assuming neither belongs to a nearest-neighbor pair: a site will be created (respectively, destroyed) if this parity is even (respectively, odd). In turn, either type of interaction will reverse the parity of a gap containing the newly created or destroyed site. Thus, we can reduce the dynamics mod 2, and study the evolution of the gap parities

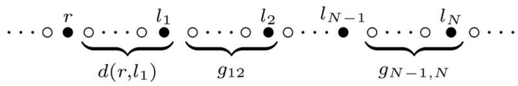

FIG. 2. Notation for interparticle gaps and separations, one right mover and N left movers. p is the parity of $d(r,l_1)$.

according to these simple rules. The presence of pairs can invalidate the analysis when the exclusion rule prevents destruction events which would otherwise occur. The corresponding parity reversals then do not take place.

Thus, our plan for analyzing the dynamics is to assume first that no pairs are present, and no pairs form during evolution. Analysis via parity then leads to very simple and general results, which are valid if they are self-consistent, that is, if they do not lead to the formation of pairs. We then consider the excluded cases in which pairs are present initially or form later. The pairs can violate the parity "selection rules," causing transitions between the types of behavior observed when those rules hold. At present, we can only analyze the effects of pairs in a laborious case-by-case manner with small numbers of particles. An important task for the future is to develop appropriate concepts for a general analysis of the effects of pairs, which might be thought of as quasiparticles in the system.

To introduce the technique of parity analysis, consider the case of one right mover versus a string of $N \equiv N_L$ left movers, without pairs. We work in the rest frame of the left movers, so the right-moving particle advects by two sites per time step, preserving parities. We label the left movers from 1 to N in the order that the right mover will encounter them, and describe the system by the gaps $g_{12}, g_{23}, \ldots, g_{N1}$ between successive left movers as well as the parity p of the separation $d(r,l_1)$ between the right mover and the first left mover (see Fig. 2). (The term "gap" always refers to the distance between consecutive left movers, or consecutive right movers; we use the term "separation" for the shrinking distance between a left mover and right mover before they scatter.) This data is preserved by advection, and changes as follows under scattering:

$$[p;g_{12},g_{23},\ldots,g_{N1}] \rightarrow [|p+g_{12}|;g_{23},\ldots,g_{N1},g_{12}+(-1)^p],$$

where $|x|$ is the parity of x. The gaps are always listed in the order that the right mover will encounter them.

Let the right mover make a complete circuit, passing all N left movers and returning to face the first again. (Note: this complete circuit in the rest frame is only a half circuit in the lattice-fixed frame. In that frame, the right mover passes each left mover twice as every particle makes a complete circuit of the lattice.) Each gap parity $g_{i,i+1}$ has been reversed, and p has changed by the sum of the original gap parities, which is the parity of the original lattice length L. (It may be surprising that the parity of L prior to the complete circuit determines whether the right mover returns to its original position relative to the first left mover or is offset from it by one site. The point is that sites which may be created or destroyed during this circuit do not increase or decrease the distance the right mover must advect to finish the circuit. This is because the right mover is carried forward or backward with the newly created or destroyed site as part of the scattering step. Changes in L during this circuit take effect at the next circuit.) The parity of the lattice length itself has changed by $|N|$ due to the N scattering events. Now let the right mover make another circuit. This restores all the original gap parities $g_{i,i+1}$, but not necessarily p, which differs from its original value by $|2L+N|=|N|$.

Supposing first that N is even, the system is parity-periodic with period two circuits, that is, $2N$ interactions. This implies that further circuits will repeat the same pattern of creation and destruction interactions as the first two circuits. However, the actual gap lengths are not periodic, because the interactions of the second circuit do not undo the effects of the first. If, for example, L is odd, it is easy to see that g_{12} is unchanged after two circuits while g_{23} has changed by ± 2. Indeed, regardless of $|L|$, alternate gaps have changed by $0, \pm 2, 0, \pm 2, \ldots$, after two circuits. The system either grows forever (if all signs are $+$) or a pair eventually forms and invalidates the analysis.

Suppose next that N is odd. After two circuits p has reversed parity while all other gaps and L have their original parities. Therefore each scattering event of circuit three will be opposite in character to the corresponding event of circuit one (site creation replacing site destruction and vice-versa), and likewise for circuits 4 and 2. After four circuits, every gap has its original length and p has its original parity: the evolution is truly periodic with period four circuits, or $4N$ interactions. During these four circuits, each gap can change by at most ± 2 from its initial size. Therefore, $g_{i,i+1} > 3$ is sufficient to prevent the formation of pairs and render this behavior self-consistent. We have thus established the existence of a large class of periodic solutions.

Note that there are 2^{N+1} possible parity states, given by the parities of the N gaps and of p. Since this number is finite, periodic evolution of the parities is inevitable. Since the period, $2N$ or $4N$, only grows linearly with N, the 2^{N+1} states clearly belong to a large number of disjoint parity orbits when N is large. Some examples with N small are given in the next section.

With a mild additional assumption this analysis can be extended to the general case of N_R right movers versus N_L left movers, of course assuming the absence of pairs. Provided that no right mover is very near a left mover in the initial state, we can again consider a complete circuit in which every right mover passes every left mover exactly once. Each right mover interacts N_L times, making $N_R N_L$ interactions in all. In the rest frame of the left movers, it is clear that each gap between consecutive left movers changes parity by $|N_R|$ during one circuit of the right movers. Similarly, each gap between consecutive right movers changes parity by $|N_L|$. There may also be a parity change in the separation between a chosen left mover and a chosen right mover after one circuit. It suffices to compute this for one such choice, because the others are then determined by the known gap changes. However, this "offset parity" depends on the exact configuration of the particles around the ring. We will always work in the rest frame of the left movers, and compute the offset between a chosen right mover and the first left mover it will encounter.

For example, suppose that initially one arc of the ring contains all the left movers and no right movers, and a disjoint arc contains all the right movers. Consider the leading right mover. As it passes the left movers, it is carried along with the creation and destruction events, and its offset parity after one cycle is simply $|L|$. Contrast this with an initial configuration in which $N_L = N_R = N$ and the left and right movers alternate around the ring. Choose a "leading" right mover arbitrarily. Number the left movers in the order that this right mover will encounter them. It passes the first and traverses the gap g_{12}, but the next gap has already been changed to $g_{23} \pm 1$ by the interaction with the right mover ahead of the chosen one. Similarly the next gap will be $g_{34} \pm 1 \pm 1$ (independent signs) when the chosen particle gets there, and the offset has the parity of $|L+1+2+\cdots+(N-1)| = |L+\frac{1}{2}N(N-1)|$. In general, the offset is L plus the number of interactions with left movers which occur before the chosen right mover reaches them. The latter contribution cancels out (mod 2) after two circuits, when the offset for any particle will be $|2L+N_L N_R| = |N_L N_R|$.

Now we analyze the various parity combinations in detail. Suppose first that both N_R and N_L are even. Then the parity of every gap is unchanged after one circuit. If the offset parity for some right mover is also even, this means that the separation of this right mover from the first left mover it will encounter is unchanged in parity after one circuit. Adding suitable right-right or left-left gaps gives this separation for any other right mover, which is therefore also unchanged in parity. Therefore the offset is even for every particle, and the pattern of interactions at every successive circuit is identical. The system either grows indefinitely or eventually forms a pair and invalidates the analysis. If, however, the offset for some (hence every) particle was odd, then the interactions of the second circuit undo the effects of the first, and the system is truly periodic with period two circuits or $2N_R N_L$ interactions. No gap changes by more than $\max(N_R, N_L)$ during the evolution.

Next suppose that N_R and N_L are both odd. Now every gap changes parity in a circuit, as does the lattice length. After two circuits, the gap parities have their original values, but the offset is odd. As in the case of 1 versus odd N above, each scattering event of circuit three is opposite in character to the corresponding event of circuit one, and likewise for circuits four and two. This results in truly periodic solutions with period four circuits or $4N_R N_L$ interactions.

Finally, suppose N_R is odd and N_L is even (the opposite case being the same by symmetry). After one cycle the left-left gap parities are reversed, while the parities of the right-right gaps and the lattice length are unchanged. As in the case of 1 versus even N above, the interactions of the next cycle cancel those of the first for every other left-left gap, but augment those of the first for the remaining left-left gaps. The result is either net growth or pair formation.

To summarize, in the absence of pairs periodic solutions are quite generic in the cases N_R, N_L both odd, and both even with offset parity odd. Now consider any solution which grows indefinitely. Because the gap and offset parities repeat after a certain number of circuits, and determine the pattern of interactions and thus the net number of sites created, the growth is characterized by some average number of sites created per circuit. Let k be this number, and $L(t)$ be the lattice length at time t. Since a circuit takes $L/2$ time steps, in the limit of continuous time we have the differential equation

$$\frac{dL}{dt} = \frac{2k}{L},$$

with asymptotic solution $L \sim 2\sqrt{kt}$. The $\sqrt{t}$ growth observed in simulations thus simply reflects constant average growth per circuit.

It is tempting to claim that this reasoning is completely general, applying even if nearest-neighbor pairs form. The argument would be that the pattern of interactions is still determined by a finite set of data, namely, the parities of all gaps and an offset, and a list of which particles are paired. As this finite set of data changes, it must eventually return to a former state, from which point the pattern of interactions will be periodic. There will be a net number of sites created per period, leading again to the $\sqrt{t}$ growth, on a sufficiently long time scale. However, this finite set of data is in fact insufficient to determine the sequence of interactions, because one needs the actual gap sizes, not just their parities, to predict when a new pair will form. Thus, at present we cannot prove that every nonperiodic solution grows according to the $\sqrt{t}$ law.

III. FEW-PARTICLE SYSTEMS

In this section (and the appendices) we will completely solve the dynamics for all initial states containing at most four particles. By symmetry we may assume at most two are right movers; for the moment we consider a single right mover with up to three left movers. We describe the states of the system at time t by the parity p of the separation between the right mover and the left mover it will encounter next, and by the gaps $g_{i,i+1}$ between left movers i and $i+1$ (where the indices $i, i+1$ are taken modulo N_L), as in Fig. 2. After every interaction we give the new state.

One against N_L

Example 2: One against two. The first nontrivial case is one right mover against two left movers. Here, the exclusion rule can apply (nearest-neighbor pairs suppressing the destruction of sites).

We describe the possible states as follows: $[p; g_{i,i+1}, g_{i+1,i+2}; |L|]$, where p is the parity of the separation $d(r, l_i)$ between r and the closest left mover l_i ($i=1,2$), and $|L|$ is the parity of the lattice length. Note that $|L| = |g_{12}+g_{21}|$. For convenience we display parity by using bold letters $\mathbf{g_{i,i+1}}$ if and only if the gap $g_{i,i+1}$ is odd.

There are eight possible combinations of the parities of $d(r, l_i)$, $g_{i,i+1}$, $g_{i+1,i+2}$. We start with the two states (1) and (i), assuming at first that no pairs occur (see below).

(1)	$[1;\mathbf{g_{12}},g_{21};1]$	
		$\downarrow -1$
(2)	$[0;g_{21},g_{12}-1;0]$	
		$\downarrow +1$
(3)	$[0;g_{12}-1,\mathbf{g_{21}+1};1]$	
		$\downarrow +1$
(4)	$[0;\mathbf{g_{21}+1},\mathbf{g_{12}};0]$	
		$\downarrow +1$
(5)	$[1;\mathbf{g_{12}},g_{21}+2;1]$	

(i)	$[1;g_{12},g_{21};0]$	
		$\downarrow -1$
(ii)	$[1;g_{21},\mathbf{g_{12}-1};1]$	
		$\downarrow -1$
(iii)	$[1;\mathbf{g_{12}-1},\mathbf{g_{21}}-1;0]$	
		$\downarrow -1$
(iv)	$[0;\mathbf{g_{21}-1},g_{12}-2;1]$	
		$\downarrow +1$
(v)	$[1;g_{12}-2,g_{21};0]$	

In this chart, the quantities g_{12} and g_{21} denote the *initial* values of these gaps; current values are indicated by position within the brackets. For example, line (3) indicates that the right mover is about to encounter l_1, the gap from l_1 rightward to l_2 is currently $g_{12}-1$, and the gap from l_2 rightward to l_1 is $g_{21}+1$. Observe that the eight states (1)–(4) and (i)–(iv) cover all the parity configurations. Thus if no pairs occur, these eight states describe all possible behaviors of one particle against two.

Consider the first example, states (1)–(5). Note that the parities of state (5) are a repetition of the parities of state (1). There is one gap that has grown by two, $g_{21}\mapsto g_{21}+2$, while the other gap remained constant. So this is a parity-periodic growing orbit as long as no pairs appear, consistent with the analysis of one against even N in Sec. II. The only places where a pair can form are in (1), (5), (9), and so on. The restriction $g_{12}\geqslant 3$ prevents the formation of pairs. In particular, we have a growing orbit if and only if $g_{12}\geqslant 3$. The growth rate is as $\sqrt{t}$ as discussed previously.

The second example describes a shrinking system. After four interactions we get back to the same configuration of parities but with a shorter lattice. Two sites have been eliminated. It is clear that eventually pairs will form and alter the evolution in the states (iii), (vii), (xi), etc., where we have an odd distance of r to l_1. Although we will consider the effects of these pairs below, the eventual fate of this system can be determined by the following time-reversal argument. Running time backward from the initial state, we would obviously see this system grow, with no pair formation. By the Evolution Theorem, this system must eventually grow in the future as well.

To complete the analysis of one right mover against two left movers, we now describe what happens if pairs form. In that case $g_{12}=1$, and we can assume that the separation $d(r,l_1)$ is odd, since otherwise the existence of the pair does not affect the evolution. Since we have the choice of the parity of g_{21}, there are two types of states with a pair:

(A)		(B)	
$[1;\mathbf{g_{12}=1},g_{21};1]$		$[1;\mathbf{g_{12}=1},\mathbf{g_{21}};0]$	
	$\downarrow \mathbf{+1}$		$\downarrow \mathbf{+1}$
$[0;\mathbf{g_{12}=1},\mathbf{g_{21}+1};0]$		$[1;\mathbf{g_{12}=1},g_{21}+1;1]$	

In the above evolution, the boldface values of $\Delta L=+\mathbf{1}$ indicate steps at which the evolution was altered by the presence of the pair. Note that (A) has evolved into step (4). Since in (A) we started with $g_{21}\geqslant 2$, we get $g_{21}+1\geqslant 3$ and remain in the growing orbit. Case (B) has evolved into (A) after one step, so both (A) and (B) evolve to the growing orbit. The pair breaks, and the gap between these neighbors subsequently grows.

Now let r face a sequence of left movers, $l_1,\ldots,l_{N_L}$, the initial state being $[p;g_{12},g_{23},\ldots,g_{N_L 1}]$. If l_1 and l_2 do not form a pair, then the state following the first interaction is as given in the previous section, namely,

$$[|p+g_{12}|;g_{23},\ldots,g_{N_L 1},g_{12}+(-1)^p].$$

If particles 1 and 2 do form a pair, this changes the outcome iff $p=1$. In that case, r does not interact with l_1, and following the interaction with l_2 the state will be

$$[|g_{23}|;g_{34},\ldots,g_{N_L 1},g_{12},g_{23}+1].$$

Recall that the interaction between r and a single pair can be conceptualized as follows. If the separation $d(r,l_1)$ to the leading member of the pair is odd, the right mover lands on the trailing member of the pair. The pair remains intact and a site is created behind it, so that $\Delta L=1$. In this case the pair behaves as a unit like a single left mover. If the separation $d(r,l_1)$ is even, the right mover lands on the leading member of the pair. A site is created between the left movers, breaking the pair, and another site is destroyed behind the pair, resulting in $\Delta L=0$. More globally, under the right conditions a pair may persist indefinitely, behaving like a quasiparticle in the system, as we will see later. Or a pair might break and reform repeatedly. Notice that, in contrast, three left movers on adjacent sites cannot form a stable "triple;" it will necessarily be broken by the next interaction with a right mover.

Example 3: One against three. Let us turn to the case of one right mover against three left movers. Here we describe a state by the parity p of $d(r,l_1)$ (assuming r faces l_1 next), by g_{12}, g_{23}, g_{31} and (for convenience, although it is redundant) the parity $|L|$ of the length of the lattice. There are sixteen possible parity configurations for the gaps and separation p.

We obtain two periodic orbits of length 12 under the assumption that no pairs form, consistent with the analysis of one against odd N in Sec. II. We give them here:

(1) $[1;g_{12},g_{23},g_{31};0]$

$\downarrow -1$

(2) $[1;g_{23},g_{31},\mathbf{g_{12}-1};1]$

$\downarrow -1$

(3) $[1;g_{31},\mathbf{g_{12}-1},\mathbf{g_{23}-1};0]$

$\downarrow -1$

(4) $[1;\mathbf{g_{12}-1},\mathbf{g_{23}-1},\mathbf{g_{31}-1};1]$

$\downarrow -1$

(5) $[0;\mathbf{g_{23}-1},\mathbf{g_{31}-1},g_{12}-2;0]$

$\downarrow +1$

(6) $[1;\mathbf{g_{31}-1},g_{12}-2,g_{23};1]$

$\downarrow -1$

(7) $[0;g_{12}-2,g_{23},g_{31}-2;0]$

$\downarrow +1$

(8) $[0;g_{23},g_{31}-2,\mathbf{g_{12}-1};1]$

$\downarrow +1$

(9) $[0;g_{31}-2,\mathbf{g_{12}-1},\mathbf{g_{23}+1};0]$

$\downarrow +1$

(10) $[0;\mathbf{g_{12}-1},\mathbf{g_{23}+1},\mathbf{g_{31}-1};1]$

$\downarrow +1$

(11) $[1;\mathbf{g_{23}+1},\mathbf{g_{31}-1},g_{12};0]$

$\downarrow -1$

(12) $[0;\mathbf{g_{31}-1},g_{12},g_{23};1]$

$\downarrow +1$

(1) $[1;g_{12},g_{23},g_{31};0]$

(a) $[1;g_{12},\mathbf{g_{23}},g_{31};1]$

$\downarrow -1$

(b) $[1;\mathbf{g_{23}},g_{31},\mathbf{g_{12}-1};0]$

$\downarrow -1$

(c) $[0;g_{31},\mathbf{g_{12}-1},g_{23}-1;1]$

$\downarrow +1$

(d) $[0;\mathbf{g_{12}-1},g_{23}-1,\mathbf{g_{31}+1};0]$

$\downarrow +1$

(e) $[1;g_{23}-1,\mathbf{g_{31}+1},g_{12};1]$

$\downarrow -1$

(f) $[1;\mathbf{g_{31}+1},g_{12},\mathbf{g_{23}-2};0]$

$\downarrow -1$

(g) $[0;g_{12},\mathbf{g_{23}-2},g_{31};1]$

$\downarrow +1$

(h) $[0;\mathbf{g_{23}-2},g_{31},\mathbf{g_{12}+1};0]$

$\downarrow +1$

(i) $[1;g_{31},\mathbf{g_{12}+1},g_{23}-1;1]$

$\downarrow -1$

(j) $[1;\mathbf{g_{12}+1},g_{23}-1,\mathbf{g_{31}-1};0]$

$\downarrow -1$

(k) $[0;g_{23}-1,\mathbf{g_{31}-1},g_{12};1]$

$\downarrow +1$

(l) $[0;\mathbf{g_{31}-1},g_{12},\mathbf{g_{23}};0]$

$\downarrow +1$

(a) $[1;g_{12},\mathbf{g_{23}},g_{31};1]$

The first system is free of pairs if and only if $g_{12}\geqslant 4$, $g_{23}\geqslant 2$, and $g_{31}\geqslant 4$: the only cases where the exclusion rule can apply are in steps (4) if $g_{12}-1=1$, in (6) if $g_{31}-1=1$, and in (11) if $g_{23}+1=1$. [Of course, this would literally imply $g_{23}=0$, which is not possible by the exclusion principle. What is meant is that an initial state with the parities of line (11) would contain a pair if the entry $g_{23}+1$ were 1 instead.]

The second system is free of pairs if and only if $g_{12}\geqslant 2$, $g_{23}\geqslant 3$, $g_{31}\geqslant 2$. The only cases where the exclusion rule can apply are in steps (b) if $g_{23}=1$, in (f) if $g_{31}+1=1$, and in (j) if $g_{12}+1=1$.

Note that all sixteen parity configurations appear in the two orbits. The first orbit covers 12 parity configurations. In the second orbit, the parity configurations repeat after 4 steps, but the actual gap sizes have period 12.

What is left is to understand the cases where pairs form (and thus the exclusion rule applies). In other words we have to study the systems with $d(r,l_1)$ odd, $g_{12}=1$ and all possible parities of g_{23},g_{31}. We give them labels as follows:

parities of (g_{23},g_{31})	label
[0,0]	(A)
[0,1]	(B)
[1,0]	(C)
[1,1]	(D)

States of types (B) and (C) belong to a single orbit. The pair stays intact and acts as a permanent quasiparticle, and the lattice grows exactly as in the 1 vs 2 case:

(C) $[1;\mathbf{g_{12}=1},\mathbf{g_{23}},g_{31};0]$

$\downarrow \mathbf{+1}$

$[1;g_{31},\mathbf{g_{12}=1},g_{23}+1;1]$

$\downarrow -1$

(B) $[1;\mathbf{g_{12}=1},g_{23}+1,\mathbf{g_{31}-1};0]$

$\downarrow \mathbf{+1}$

$[0;\mathbf{g_{31}-1},\mathbf{g_{12}=1},\mathbf{g_{23}+2};1]$

$\downarrow +1$

(C) $[1;\mathbf{g_{12}=1},\mathbf{g_{23}+2},g_{31};0]$

To understand why the pair is permanent in this example, recall from Sec. II or example 2 the simpler case of one right mover against two left movers, with p and $|L|$ even. In this system, p and $|L|$ remain even forever and ΔL alternates between 0 and +2 on successive circuits. Now convert the leading left mover into a pair by adding a third left mover immediately in front of it. p is now odd and the new left mover is completely inert, as the right mover always lands on the trailing (original) member of this pair. The pair is permanent and the evolution is unaffected by the third left mover.

States of types (A,D) belong to a single orbit, and the details of their evolution are given in Appendix A. This time the pair is "semipermanent," repeatedly breaking but immediately reforming.

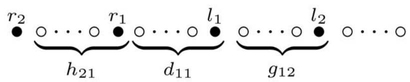

FIG. 3. Notation for two right movers vs. two left movers.

Example 4: Two against two. This case needs more information: now there are two right movers and they interact in a row. We have to adapt the notation and keep track of the positions of the right movers (see Fig. 3).

Suppose r_1 interacts first with l_1. Then there are several possibilities.

(i) Particle r_1 interacts next with l_2, then r_2 interacts with l_1.

(ii) There is an interlaced pattern: particle r_2 interacts with l_1, then r_1 interacts with l_2, then r_2 with l_2.

(iii) Particles r_1 and l_2 interact at the same time step as r_2 with l_1.

It can be checked that it does not matter which interaction is first (as long as no pairs occur). This is a reflection of the locality of the scattering.

Figure 3 shows a "noninterlaced" state in which there is no left mover between the two right movers, and vice versa. We can always assume such an initial state, unless the four particles are located symmetrically around the ring: right movers diametrically opposite one another, left movers likewise along a perpendicular diameter. It is easy to see that these exceptional initial states evolve as periodic orbits, and we will not consider them further.

The complete set of examples of the evolution of all possible initial states appears in Appendix B. Here we merely summarize the main features. States without pairs follow the analysis of Sec. II for N_R, N_L even. When the offset is odd, the solutions are periodic with period 8 interactions. For even offset the lattice grows, or shrinks until a pair forms. When pairs are present, we find that in all cases they eventually break and the lattice grows as $\sqrt{t}$.

IV. CONCLUSIONS

In this paper we have studied the lattice gas model with dynamical geometry introduced by Hasslacher and Meyer [4]. We first gave a general discussion of the pair-free evolution, establishing the importance of parity and the existence of many periodic orbits. The $\sqrt{t}$ growth of the lattice length observed in simulations is expected whenever the length grows by a constant amount, on average, as the particles complete one circuit. This is the case without pairs, and it is plausible that it is the general asymptotic behavior, but we have no proof of this. At present we can include the effects of pairs only by an exhaustive case-by-case analysis, which we carried out for systems of at most four particles. The pairs can form permanent "bound states," and it seems promising to view them as quasiparticles. With an effective description of these quasiparticles, we may hope to solve the dynamics of this model completely.

Although the microscopic dynamics of this model is reversible, it exhibits macroscopic "irreversibility" as the length grows for "most" states. It might be interesting to quantify what fraction of initial states having given N_L,N_R,L ultimately grow.

The natural conjecture regarding the asymptotics of the model is that all nonperiodic solutions grow as $\sqrt{t}$, reflecting a constant average growth per circuit. Is this really true? Perhaps there are solutions with a characteristic time scale much longer than one circuit. For example, imagine a solution in which the length initially has average growth zero per circuit. A randomly chosen right-right gap would have size of order L/N_R and could shrink to form a pair in a time $\sim L^2/N_R N_L$. If this pair alters the evolution to produce constant average growth on this time scale, then $dL/dt \sim 1/L^2$ leads to $L \sim t^{1/3}$. Are there solutions with this behavior? Do the fluctuations in a solution growing as $\sqrt{t}$ include intervals when the growth is as $t^{1/3}$, correlated with the formation and destruction of pairs?

Finally, it should be straightforward to quantize this model lattice gas, generalizing Refs. [7,8]. The Hilbert space would be the direct sum $H = \oplus_{L=1}^{\infty} H_L$ of the Hilbert spaces for lattices of all fixed lengths L. Advection occurs within each H_L, but scattering causes transitions between them. The quantized model would be similar to that of Ref. [2], but with evolution in discrete rather than continuous time. It is possible that the model, at least with few particles, is solvable by Bethe ansatz or other methods.

ACKNOWLEDGMENTS

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APPENDIX A: THE ONE AGAINST THREE SYSTEM WITH PAIRS

We continue example 3 with the evolution of state (A), which appears below. Note that after eight interactions, we have a state of type (D), so type (D) is contained in the same orbit. As long as no other pairs appear, type (A) forms a parity-periodic orbit of length 16. $\Delta L = -2$ after one such orbit: the gap between particles l_2,l_3 decreases by 2. Pairs can appear in state (d) if $g_{23}+1=1$, in state (j) if $g_{31}+1=1$, and in state (o) if $g_{23}-1=1$.

Suppose $g_{31}+1=1$ in (j). This is a pair as in type (C), so from here on, the orbit is growing.

Let $g_{23}-2k+1=1$. The change in the pattern occurs in the kth run through the orbit (a)–(p). For $k=0$ state (d) contains a pair of type (C). For $k>0$ state (o) contains a pair of type (D), [1;1,1,1] with first and third gap length equal to one. This state evolves to state (j) with first gap of length one: a type (C) pair has formed. We already know that type (C) belongs to a growing orbit.

So systems of type (A) and (D) produce a parity-periodic orbit of decreasing length. As soon as a second pair forms, we observe a transition via (D) to the growing orbit of type (B).

(a) $[1;\mathbf{g_{12}=1},g_{23},g_{31};1]$

$\downarrow\mathbf{+1}$

(b) $[0;g_{31},\mathbf{g_{12}=1},\mathbf{g_{23}+1};0]$

$\downarrow+1$

(c) $[0;\mathbf{g_{12}=1},\mathbf{g_{23}+1},\mathbf{g_{31}+1};1]$

$\downarrow+1$

(d) $[1;\mathbf{g_{23}+1},\mathbf{g_{31}+1},g_{12}+1;0]$

$\downarrow-1$

(e) $[0;\mathbf{g_{31}+1},g_{12}+1,g_{23};1]$

$\downarrow+1$

(f) $[1;g_{12}+1,g_{23},g_{31}+2;0]$

$\downarrow-1$

(g) $[1;g_{23},g_{31}+2,\mathbf{g_{12}=1};1]$

$\downarrow-1$

(h) $[1;g_{31}+2,\mathbf{g_{12}=1},\mathbf{g_{23}-1};0]$

$\downarrow-1$

(i) $[1;\mathbf{g_{12}=1},\mathbf{g_{23}-1},\mathbf{g_{31}+1};1]$

$\downarrow\mathbf{+1}$

(j) $[1;\mathbf{g_{31}+1},\mathbf{g_{12}=1},g_{23};0]$

$\downarrow-1$

(k) $[0;\mathbf{g_{12}=1},g_{23},g_{31};1]$

$\downarrow+1$

(l) $[1;g_{23},g_{31},g_{12}+1;0]$

$\downarrow-1$

(m) $[1;g_{31},g_{12}+1,\mathbf{g_{23}-1};1]$

$\downarrow-1$

(n) $[1;g_{12}+1,\mathbf{g_{23}-1},\mathbf{g_{31}-1};0]$

$\downarrow-1$

(o) $[1;\mathbf{g_{23}-1},\mathbf{g_{31}-1},\mathbf{g_{12}=1};1]$

$\downarrow-1$

(p) $[0;\mathbf{g_{31}-1},\mathbf{g_{12}=1},g_{23}-2;0]$

$\downarrow+1$

(a′) $[1;\mathbf{g_{12}=1},g_{23}-2,g_{31};1]$

APPENDIX B: DETAILS OF THE TWO AGAINST TWO SYSTEM

We start by looking at the cases where no pairs appear. In order to describe the evolution of the system, we give pairs of 4-tuples $[d_{11};g_{12},g_{21};|L|]$ $[d_{21};g_{12},g_{21};|L|]$.

In contrast to the previous notation, we give the actual separation between the right mover r_i and the next left mover l_j that r_i faces. Let $h_{21}\coloneqq d(r_2,r_1)$ be the gap between the two right movers. As before, bold font is used to denote odd length. Furthermore, we add an arrow $\pm1\downarrow$ to show the change of the length of the lattice. The arrows also indicate which pair is interacting.

Note that $d_{21}=d_{11}+h_{21}$. We know that the order of the interactions does not matter. Thus in the example below we can fix the order of interactions by assuming $h_{21}\leqslant g_{12}-2$ in (2) and $h_{21}+1\leqslant g_{21}-2$ in (4).

Example 1. We start with a system where g_{12} is even, g_{21} odd (i.e., the lattice has odd length). Also let h_{21} and d_{11} be even. Recall that notation such as g_{12} below denotes this gap in the *initial* state (1); as the gaps change during evolution their current values are indicated by their positions:

	Particle r_1		Particle r_2	
(1)	$[d_{11};g_{12},\mathbf{g_{21}};1]$		$[h_{21}+d_{11};g_{12},\mathbf{g_{21}};1]$	
		$\downarrow+1$		
(2)	$[g_{12};\mathbf{g_{21}},\mathbf{g_{12}+1};0]$		$[h_{21};\mathbf{g_{12}+1},\mathbf{g_{21}};0]$	
				$\downarrow+1$
(3)	$[g_{12}-h_{21};\mathbf{g_{21}},g_{12}+2;1]$		$[\mathbf{g_{12}+1};g_{12}+2,\mathbf{g_{21}};1]$	
		$\downarrow+1$		
(4)	$[\mathbf{g_{21}};g_{12}+2,g_{21}+1;0]$		$[\mathbf{h_{21}+1};g_{21}+1,g_{12}+2;0]$	
				$\downarrow-1$
(5)	$[\mathbf{g_{21}-h_{21}-2};g_{12}+2,\mathbf{g_{21}};1]$		$[\mathbf{g_{21}};g_{12}+2,\mathbf{g_{21}};1]$	
		$\downarrow-1$		
(6)	$[\mathbf{g_{12}+1};\mathbf{g_{21}},\mathbf{g_{12}+1};0]$		$[\mathbf{h_{21}+1};\mathbf{g_{12}+1},\mathbf{g_{21}};0]$	
				$\downarrow-1$
(7)	$[\mathbf{g_{12}-h_{21}-1};\mathbf{g_{21}},g_{12};1]$		$[g_{12};\mathbf{g_{21}},g_{12};1]$	
		$\downarrow-1$		
(8)	$[g_{21}-1;g_{12},g_{21}-1;0]$		$[h_{21};g_{21}-1,g_{12};0]$	
				$\downarrow+1$
(1)′	$[g_{21}-h_{21}-1;g_{12},\mathbf{g_{21}};1]$		$[g_{21}-1;g_{12},\mathbf{g_{21}};1]$	

The assumption $g_{21}\geqslant3$ (i.e., the gap between l_2 and l_1 is at least three) ensures that the exclusion rule does not apply in step (7).

There are 16 possible parity combinations for $h_{21},d_{11},g_{12},g_{21}$. In that language, the parities that occur in example 1 above are [0,0,0,1] in (1), [1,0,1,0] in (3), [0,1,0,1] in (5), and (1,1,1,0) in (7).

Example 2. Next we assume d_{11} to be even, h_{21},g_{21} even and g_{12} odd:

	Particle r_1		Particle r_2	
(1)	$[d_{11};\mathbf{g_{12}},g_{21};1]$		$[h_{21}+d_{11};\mathbf{g_{12}},g_{21};1]$	
		$\downarrow+1$		
(2)	$[\mathbf{g_{12}};g_{21},g_{12}+1;0]$		$[h_{21};g_{12}+1,g_{21};0]$	
				$\downarrow+1$
(3)	$[\mathbf{g_{12}-h_{21}};g_{21},\mathbf{g_{12}+2};1]$		$[g_{12}+1;g_{21},\mathbf{g_{12}+2};1]$	
		$\downarrow-1$		
(4)	$[\mathbf{g_{21}-1};\mathbf{g_{12}+2},\mathbf{g_{21}-1};0]$		$[h_{21};\mathbf{g_{21}-1},\mathbf{g_{12}+2};0]$	
				$\downarrow+1$
(5)	$[\mathbf{g_{21}-h_{21}-1};\mathbf{g_{12}+2},g_{21};1]$		$[\mathbf{g_{21}-1};\mathbf{g_{12}+2},g_{21};1]$	
		$\downarrow-1$		
(6)	$[g_{12}+1;g_{21},g_{12}+1;0]$		$[\mathbf{h_{21}-1};g_{12}+1,g_{21};0]$	
				$\downarrow-1$
(7)	$[g_{12}-h_{21}+1;g_{21},\mathbf{g_{12}};1]$		$[\mathbf{g}_{12};g_{21},\mathbf{g_{12}};1]$	
		$\downarrow+1$		
(8)	$[g_{21};\mathbf{g_{12}},\mathbf{g}_{21}+\mathbf{1};0]$		$[\mathbf{h}_{21}-\mathbf{1};\mathbf{g_{21}+1},\mathbf{g_{12}};0]$	
				$\downarrow-1$
(1)′	$[g_{21}-h_{21};\mathbf{g_{12}},g_{21};1]$		$[g_{21};\mathbf{g_{12}},g_{21};1]$	

This is again a periodic orbit. The only situations where the exclusion rule can apply are in state (5) if $g_{12}+2=1$ [i.e., $d(l_1,l_2)=1$] or in state (3) if $g_{12}+1-(g_{12}-h_{21})=h_{21}+1=1$ [i.e., $d(r_2,r_1)=1$]. Note that the odd numbered states are the interlaced ones. The parity configurations of the distances $d(r_{i+1},r_i)$, $d(r_i,l_j)$, $d(l_j,l_{j+1})$, $d(l_{j+1},l_{j+2})$ are [0,0,1,0] in (1), [1,1,0,1] in (3), [0,1,1,0] in (5) and [1,1,0,1] in (7). Now we describe the systems where the length of the lattice is increasing.

Example 3. Let h_{21} and d_{11} be even, g_{12} and g_{21} odd:

	Particle r_1		Particle r_2	
(1)	$[d_{11};\mathbf{g_{12}},\mathbf{g_{21}};0]$		$[h_{21}+d_{11};\mathbf{g_{12}},\mathbf{g_{21}};0]$	
		$\downarrow+1$		
(2)	$[\mathbf{g_{12}};\mathbf{g_{21}},g_{12}+1;1]$		$[h_{21};g_{12}+1,\mathbf{g_{21}};1]$	
				$\downarrow+1$
(3)	$[\mathbf{g_{12}-h_{21}};\mathbf{g_{21}},\mathbf{g_{12}+2};0]$		$[g_{12}+1;\mathbf{g_{21}},\mathbf{g_{12}+2};0]$	
		$\downarrow-1$		
(4)	$[g_{21}-1;\mathbf{g_{12}+2},g_{21}-1;1]$		$[h_{21};g_{21}-1,\mathbf{g_{12}+2};1]$	
				$\downarrow+1$
(1)′	$[g_{21}-h_{21}-1;\mathbf{g_{12}+2},\mathbf{g_{21}};0]$		$[g_{21}-1;\mathbf{g_{12}+2},\mathbf{g_{21}};0]$	

After four interactions we return to the initial parity configuration, with $\Delta L=2$ [the gap $d(l_1,l_2)$ grows by 2]. In the noninterlaced states we have the following parity configurations of the distances $d(r_{i+1},r_i)$, $d(r_i,l_j)$, $d(l_j,l_{j+1})$, $d(l_{j+1},l_{j+2})$: [0,0,1,1] in (1) and [1,1,1,1] in (3).

Example 4. In this case, h_{21} is odd, d_{11} even, g_{12} and g_{21} even:

	Particle r_1		Particle r_2	
(1)	$[d_{11};g_{12},g_{21};0]$		$[\mathbf{h_{21}+d_{11}};g_{12},g_{21};0]$	
		$\downarrow+1$		
(2)	$[g_{12};g_{21},\mathbf{g_{12}+1};1]$		$[\mathbf{h_{21}};\mathbf{g_{12}+1},g_{21};1]$	
				$\downarrow-1$
(3)	$[g_{12}-h_{21}-1;g_{21},g_{12};0]$		$[g_{12};g_{21},g_{12};0]$	
		$\downarrow+1$		
(4)	$[g_{21};g_{12},\mathbf{g_{21}+1};1]$		$[h_{21}+1;\mathbf{g_{21}+1},g_{12};1]$	
				$\downarrow+1$
(1)′	$[g_{21}-h_{21}-1;g_{12},g_{21}+2;0]$		$[\mathbf{g_{21}+1};g_{12},g_{21}+2;0]$	

This is another growing lattice. After four interactions, the gap between l_2 and l_1 has grown by 2, with the other gaps unchanged. The parities are [1,0,0,0] in (1) and [0,0,0,0] in Eq. (3).

There are four remaining parity combinations. They belong to two systems with decreasing length. We describe them in the two examples below.

Example 5. Let h_{21}, g_{12} and g_{21} be even, d_{11} odd:

	Particle r_1		Particle r_2	
(1)	$[\mathbf{d_{11}};g_{12},g_{21};0]$		$[\mathbf{h_{21}+d_{11}};g_{12},g_{21};0]$	
		$\downarrow -1$		
(2)	$[\mathbf{g_{12}-1};g_{21},\mathbf{g_{12}-1};1]$		$[\mathbf{h_{21}-1};\mathbf{g_{12}-1},g_{21};1]$	
				$\downarrow -1$
(3)	$[\mathbf{g_{21}-h_{21}-1};g_{21},g_{12}-2;0]$		$[g_{12}-2;g_{21},g_{12}-2;0]$	
		$\downarrow -1$		
(4)	$[\mathbf{g_{21}-1};g_{12}-2,\mathbf{g_{21}-1};1]$		$[h_{21}-2;\mathbf{g_{21}-1},g_{12}-2;1]$	
				$\downarrow +1$
(1)′	$[\mathbf{g_{21}-h_{21}+1};g_{21}-2,g_{21};0]$		$[\mathbf{g_{21}-1};g_{12}-2,g_{21};0]$	

After four interactions we return to the same parities but with $\Delta L=-2$ (gap $g_{12}\mapsto g_{12}-2$). In other words, this system is shrinking as long as no pairs occur. A pair can form in state (3). There the distance between particles r_2 and r_1 after k cycles (1)–(4) is $g_{12}-2k-(g_{21}-h_{21}-1)$, which will eventually shrink to 1. In this case we obtain pairs as in the system with label (b) of example 8 below by switching the roles of the left movers and the right movers. The parities of this example are [0,1,0,0] in (1) and [1,1,0,0] in (3).

Example 6. Finally, let d_{11},g_{12},g_{21} be odd and h_{21} be even:

	Particle r_1		Particle r_2	
(1)	$[\mathbf{d_{11}};\mathbf{g_{12}},\mathbf{g_{21}};0]$		$[\mathbf{h_{21}+d_{11}};\mathbf{g_{12}},\mathbf{g_{21}};0]$	
		$\downarrow -1$		
(2)	$[g_{12}-1;\mathbf{g_{21}},g_{12}-1;1]$		$[\mathbf{h_{21}-1};g_{12}-1,\mathbf{g_{21}};1]$	
				$\downarrow -1$
(3)	$[g_{12}-h_{21}-1;\mathbf{g_{21}},\mathbf{g_{12}-2};0]$		$[\mathbf{g_{12}};\mathbf{g_{21}},g_{12}-2;0]$	
		$\downarrow +1$		
(4)	$[\mathbf{g_{21}};\mathbf{g_{12}-2},g_{21}+1;1]$		$[\mathbf{h_{21}-1};g_{21}+1,\mathbf{g_{12}-2};1]$	
				$\downarrow -1$
(1)′	$[\mathbf{g_{21}-h_{21}};\mathbf{g_{12}-2},\mathbf{g_{21}};0]$		$[\mathbf{g_{21}};\mathbf{g_{12}-2},\mathbf{g_{21}};0]$	

After four interactions, the length of the lattice has decreased by two (gap $g_{12}\mapsto g_{12}-2$). This is a shrinking system as long as no pairs appear. A pair appears in state (1), as soon as $g_{12}-2k=1$ [i.e., after k cycles (1)–(4)], when we obtain a pair as in the system with label b), example 8. The parities of this example are [0,1,1,1] in (1) and [1,0,1,1] in (3).

We now describe the patterns when pairs are present initially or form during the evolution (see Fig. 4). There are six different configurations with pairs. The gap d_{11} has to be odd if the pair is to affect the evolution of the system, and we assume $g_{12}=1$. Then we have the following parity

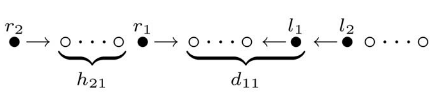

FIG. 4. Notation for two vs two with pairs.

combinations for the remaining distances:

Label	h_{21}		g_{21}
a)	0		0
b)	0		1
c)	1	with $h_{21}>1$	0
d)	1	with $h_{21}>1$	1
e)	1	with $h_{21}=1$	0
f)	1	with $h_{21}=1$	1

Example 7. [Cases (a), (d), and (c)]. We start with h_{21} and g_{21} even (i.e., a lattice of odd length). Assume that no other pair forms. The corresponding conditions are $g_{21}-(h_{21}-1)\geqslant 3$ in state (2), and $g_{21}-(g_{21}-h_{21}-1)=h_{21}+1\geqslant 3$ in state (7). We will discuss these other cases below:

	Particle r_1		Particle r_2	
(1)	$[\mathbf{d_{11}};\mathbf{g_{12}=1},g_{21};1]$		$[\mathbf{h_{21}+d_{11}};\mathbf{g_{12}=1},g_{21};1]$	
		$\downarrow\mathbf{+1}$		
(2)	$[g_{21};\mathbf{g_{12}=1},\mathbf{g_{21}+1};0]$		$[\mathbf{h_{21}-1};\mathbf{g_{12}=1},\mathbf{g_{21}+1};0]$	
				$\downarrow\mathbf{+1}$
(3)	$[g_{21}-h_{21};\mathbf{g_{12}=1},g_{21}+2;1]$		$[\mathbf{g_{21}+1};\mathbf{g_{12}=1},g_{21}+2;1]$	
		$\downarrow+1$		
(4)	$[\mathbf{g_{12}=1};g_{21}+2,g_{12}+1;0]$		$[\mathbf{h_{21}+1};g_{12}+1,g_{21}+2;0]$	
		$\downarrow-1$		
(5)	$[\mathbf{g_{21}+1};g_{12}+1,\mathbf{g_{21}+1};1]$		$[\mathbf{h_{21}-1};g_{12}+1,\mathbf{g_{21}+1};1]$	
				$\downarrow-1$
(6)	$[\mathbf{g_{21}-h_{21}+1};\mathbf{g_{12}=1},\mathbf{g_{21}+1};0]$		$[\mathbf{g_{12}=1};\mathbf{g_{21}+1},\mathbf{g_{12}=1};0]$	
				$\downarrow-1$
(7)	$[\mathbf{g_{21}-h_{21}-1};\mathbf{g_{12}=1},g_{21};1]$		$[g_{21};\mathbf{g_{12}=1},g_{21};1]$	
		$\downarrow\mathbf{+1}$		
(8)	$[g_{21};\mathbf{g_{12}=1},\mathbf{g_{21}+1};0]$		$[h_{21};\mathbf{g_{12}=1},\mathbf{g_{21}+1};0]$	
				$\downarrow+1$
(9)	$[g_{21}-h_{21};g_{21}+1,\mathbf{g_{21}+1};1]$		$[\mathbf{g_{12}=1};\mathbf{g_{21}+1},g_{12}+1;1]$	
				$\downarrow-1$
(10)	$[g_{21}-h_{21}-g_{12}-1;g_{12}+1,g_{21};0]$		$[g_{21};g_{12}+1,g_{21};0]$	

The first observation is that state (2) is as case (d) (with particles r_1, r_2 switched) by the assumption that the gap between r_1 and r_2 is at least 3. Similarly, state (7) is as case (c) The distance between r_2 and r_1 is odd and at least 3 (by the assumptions), the distance between l_2 and l_1 is even. Finally note that state (10) is as state (3) in example 4. This means that the system evolves into a growing lattice as in example 4.

Consider now other pairs of nearest neighbors. Let $g_{21}=h_{21}$ [i.e., $g_{21}-(h_{21}-1)=1$ in (2)]. Then the state (2) has the same parities as case (f): the right movers form a pair, as do the left movers, and the gap between the particles r_2 and l_1 is odd. So for $g_{21}=h_{21}$ the system will evolve as (f).

Let $h_{21}+1=1$. Then state (7) has the same parities as case (e).

Example 8 (Case b). We start with h_{21} even and g_{21} odd (i.e., a lattice of even length):

	Particle r_1		Particle r_2	
(1)	$[\mathbf{d}_{11};\mathbf{g}_{12}=\mathbf{1},\mathbf{g}_{21};0]$		$[\mathbf{h}_{21}+\mathbf{d}_{11};\mathbf{g}_{12}=\mathbf{1},\mathbf{g}_{21};0]$	
		$\downarrow\mathbf{+1}$		
(2)	$[\mathbf{g}_{21};\mathbf{g}_{21}=\mathbf{1},g_{21}+1;1]$		$[\mathbf{h}_{21}-\mathbf{1};\mathbf{g}_{12}=\mathbf{1},g_{21}+1;1]$	
				$\downarrow\mathbf{+1}$

Note that state (2) is as in (a).

It remains to discuss cases (e) and (f). These are the cases where two pairs of nearest neighbors face each other with an odd distance between them. That means we have $g_{12}=h_{21}=1$, and necessarily $g_{21}\geqslant 3$.

Example 9. Let particles r_2, r_1 be paired as well as particles l_1,l_2. Assume that g_{21} is even (i.e., lattice of odd length). So g_{21} is at least equal to 4.

	Particle r_1		Particle r_2	
(1)	$[\mathbf{d}_{11};\mathbf{g}_{12}=\mathbf{1},g_{21};1]$		$[h_{21}+d_{11};\mathbf{g}_{12}=\mathbf{1},g_{21};1]$	
		$\downarrow\mathbf{+1}$		$\downarrow\mathbf{+1}$
(2)	$[g_{21};g_{12}+1,\mathbf{g}_{21}+\mathbf{1};1]$		$[\mathbf{h}_{21}=\mathbf{1};\mathbf{g}_{21}+\mathbf{1},g_{12}+1;1]$	
				$\downarrow-1$
(3)	$[g_{21}-h_{21}-1;g_{12}+1,g_{21};0]$		$[g_{21};g_{12}+1,g_{21};0]$	

Note that state (3) has the same parities as state (3) in example 4. So the system evolves into a growing orbit as in example 4.

Example 10. Let r_2,r_1 and l_1,l_2 each be pairs, with g_{21} odd ($g_{21}\geqslant 3$). The length of the lattice is then even:

	Particle r_1		Particle r_2	
(1)	$[\mathbf{d}_{11};\mathbf{g}_{12}\mathbf{=1},\mathbf{g}_{21};0]$		$[h_{21}+d_{11};\mathbf{g}_{12}\mathbf{=1},\mathbf{g}_{21};0]$	
		$\downarrow\mathbf{+1}$		$\downarrow\mathbf{+1}$
(2)	$[\mathbf{g}_{21};g_{12}+1,g_{21}+1;0]$		$[\mathbf{h}_{21}\mathbf{=1};g_{21}+1,g_{12}+1;0]$	
				$\downarrow-1$
(3)	$[\mathbf{g}_{21}\mathbf{-h}_{21}\mathbf{-1};g_{12}+1,\mathbf{g}_{21};1]$		$[\mathbf{g}_{21};g_{12}+1,\mathbf{g}_{21};1]$	
		$\downarrow-1$		
(4)	$[\mathbf{g}_{12}\mathbf{=1};\mathbf{g}_{21},\mathbf{g}_{12}\mathbf{=1};0]$		$[\mathbf{h}_{21};\mathbf{g}_{12}\mathbf{=1},\mathbf{g}_{21};0]$	
				$\downarrow\mathbf{+1}$
(5)	$[g_{21}-h_{21};g_{21}+1,\mathbf{g}_{12},1]$		$[\mathbf{g}_{21};\mathbf{g}_{12}\mathbf{=1},g_{21}+1;1]$	
		$\downarrow+1$		
(6)	$[\mathbf{g}_{12}\mathbf{=1};g_{21}+1,g_{12}+1;0]$		$[\mathbf{h}_{21};g_{12}+1,g_{21}+1;0]$	
		$\downarrow-1$		
(7)	$[\mathbf{g}_{21};\mathbf{g}_{12}\mathbf{=1},\mathbf{g}_{21};0]$		$[\mathbf{g}_{12}\mathbf{=1};\mathbf{g}_{21},\mathbf{g}_{12}\mathbf{=1};0]$	
				$\downarrow-1$
(8)	$[\mathbf{g}_{21}\mathbf{-g}_{12}\mathbf{-1};\mathbf{g}_{12}\mathbf{=1},g_{21}-1;1]$		$[g_{21}-1;\mathbf{g}_{12}\mathbf{=1},g_{21}-1;1]$	

Note that state (8) has the same parities as state (1) in example 9. So the system will switch to that example and then to a growing orbit (as in example 4).

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